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On maximin value and policy functions in an exhaustible resource model

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This paper studies maximin paths in the context of a standard exhaustible resource model. Under the assumption that the resource is important in production, it establishes the efficiency and uniqueness of non-trivial maximin paths. It uses these results to study the nature of the maximin value and policy functions. The value function is shown to be differentiable with respect to the initial resource stock, and the derivative of the value function is related to the shadow prices associated with the maximin path starting from that resource stock. We show how maximin policy functions can be derived, using the maximin value function, and the fact that the maximin path always follows Hartwick's investment rule.

Key words maximin, exhaustible resource, Hartwick's rule, efficiency, uniqueness, value function, policy function

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1 Introduction

The (optimal) *value function* and the (optimal) *policy correspondence* have become part of the standard toolkit for economic theorists since Debreu (1959) indicated the unifying structure of constrained maximization problems by emphasizing the role of the *maximum theorem* in capturing the nature of dependence of the maximum value and the maximizers on the parameters of the constrained maximization problem.

In dynamic optimization problems, the concepts of the value function and the policy function have played an important role in the development of the theory, when the objective function incorporates *discounted utilitarianism* (DU); see, for example, Mirman and Zilcha (1975), Benveniste and Scheinkman (1979), Stokey and Lucas (1989), Araujo (1991), and Santos (1991).

For dynamic optimization problems in which other objective functions are used, the development of the theory has been less systematic. In particular, while the *maximin* objective function is quite often used, a fully developed theory of the value and policy functions for this objective function is missing from the literature.

The maximin objective has been used notably in the literature which emphasizes concepts such as sustainability of consumption levels, and intergenerational equity. These concepts are especially

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significant in the economics of non-renewable resources, where there is a natural concern that such resources are perhaps being depleted too fast by present generations. The seminal paper by Solow (1974) presents a concrete analytic model to address these issues, and this model forms the basis of the discourse in our paper.

While the model used by Solow (1974) is quite general, the analytical results on maximin paths in his model were obtained in large part in the Cobb–Douglas case. In trying to generalize his analysis, one realizes that a systematic study of *efficiency* of maximin paths is essential, and it is missing in the available literature. Building on recent work in Mitra *et al*. (2013), which deals with *existence* of maximin paths by exploiting properties of *Hartwick paths* (along which the rule of capital accumulation is to invest precisely the rents from the use of the exhaustible resource), we show, under an additional assumption[1] on the production function, that all non-trivial maximin paths are necessarily efficient. A consequence of this result is that, when the production function is capable of sustaining a positive constant consumption level forever, a non-trivial maximin path is unique.

This sets the stage for the study of the maximin value function associated with the maximin objective. We are especially interested in how this value (which depends on the initial stocks of capital and the resource) changes when the resource stock changes. That is, we are interested in measuring the gain in social welfare of an economy when there is an increase in the resource stock (because of a new discovery of a resource pool) or the loss of social welfare of an economy following the destruction of a part of its resource stock (because of a natural or man-made disaster). This can be done precisely by obtaining the derivative of the value function, and our purpose is to provide a precise estimate of the derivative in terms of (ideally) observable magnitudes like the prices associated with the maximin path. Our principal result[2] in this direction is a rigorous demonstration that the value function is differentiable (for positive resource stocks), and

$$v'(m_0) = \frac{1}{\int_0^\infty p(t)dt},$$

where m_0 is the initial resource stock, v is the value function (with the initial positive capital stock fixed, and therefore suppressed), and $(p(t))$ is the path of present value prices of the capital/consumption good (the present value price of the resource is a positive constant) associated with the maximin path starting from the initial resource stock m_0.

Our study of the maximin value function leads naturally to an examination of maximin policy functions. We show how these can be derived, using the maximin value function, and the fact that the maximin path is always a Hartwick path. As a simple application, we show how these policy functions can be calculated explicitly in the case where the production function is of the Cobb–Douglas form.

2 Preliminaries

2.1 A model of exhaustible resource use

Denote by k the stock of an augmentable capital good (which is assumed to be non-depreciating) and by r the flow of an exhaustible resource input. Denote by $F : \mathbb{R}_+^2 \to \mathbb{R}_+$ the production function for the capital/consumption good, employing k and r as inputs. The output $F(k, r)$ is used to provide a

1 This assumption conveys that the exhaustible resource is important in production in the sense that the share of the resource in total income is bounded away from zero. This is formally stated as assumption **A4** and introduced in Section 4.

2 The intuition for the formula is provided in Section 5, just before Proposition 6 and Theorem 2, which contain the formal statements and rigorous proofs.

flow of consumption, c, or to augment the capital stock through a flow of net investment, $\dot{k}$. Output $F(k, r)$ is the only source of consumption or net investment.

Throughout we will impose the following three assumptions on F (where subscripts refer to partial derivatives):

(**A1**) $F(0, r) = F(k, 0) = 0$ for $(k, r) \in \mathbb{R}^2_+$.
(**A2**) F is continuous, concave and non-decreasing on $\mathbb{R}^2_+$.
(**A3**) F is twice continuously differentiable on $\mathbb{R}^2_{++}$, with $F_1(k, r) > 0, F_2(k, r) > 0$, and $F_{22}(k, r) < 0$, for $(k, r) \in \mathbb{R}^2_{++}$.

Let $(k_0, m_0) \in \mathbb{R}^2_+$ be a vector of initial stocks of capital and resource. A *path* from (k_0, m_0) is a triplet of functions $(c(t), k(t), r(t))$, with $c(\cdot) : [0, \infty) \to \mathbb{R}_+$, $k(\cdot) : [0, \infty) \to \mathbb{R}_+$ and $r(\cdot) : [0, \infty) \to \mathbb{R}_+$, where $k(t)$ is differentiable and $(c(t), r(t))$ are continuous, and where:

$$\begin{aligned} &\text{(i) } c(t) = F(k(t), r(t)) - \dot{k}(t), \\ &\text{(ii) } k(0) = k_0, \\ &\text{(iii) } \textstyle\int_0^\infty r(t)dt \le m_0. \end{aligned} \tag{1}$$

Write $m(\cdot) : [0, \infty) \to \mathbb{R}_+$ for the associated function of remaining resource stock:

$$m(t) = m_0 - \int_0^t r(\tau)d\tau, \quad \text{for } t \ge 0. \tag{2}$$

Note that along any path $(c(t), k(t), r(t))$ from (k_0, m_0), both $k(t)$ and $m(t)$ are continuously differentiable functions of t.

A path $(c(t), k(t), r(t))$ from (k_0, m_0) is *interior* if $k(t) > 0$ and $r(t) > 0$ for all $t \ge 0$. A path $(c(t), k(t), r(t))$ from (k_0, m_0) is *egalitarian* if there is $c \ge 0$ such that $c(t) = c$ for all $t \ge 0$. A path $(c(t), k(t), r(t))$ from (k_0, m_0) is a *maximin path* if

$$\inf_{t \ge 0} c(t) \ge \inf_{t \ge 0} c'(t) \tag{3}$$

for every path $(c'(t), k'(t), r'(t))$ from (k_0, m_0). A maximin path is *non-trivial* if, in addition, $\inf_{t \ge 0} c(t) > 0$.

A path $(c'(t), k'(t), r'(t))$ from (k_0, m_0) is *efficient* if there is no path $(c(t), k(t), r(t))$ from (k_0, m_0) with $c(t) \ge c'(t)$ for all $t \ge 0$, and $c(t) > c'(t)$ for some $t \ge 0$.[3]

2.2 Hartwick's rule and a resource requirement function

A triple $(c, k, r) \gg 0$ satisfies *Hartwick's rule* if

$$F(k, r) - c = F_2(k, r)r. \tag{HaR}$$

This investment rule (due to Hartwick (1977)) plays an important role in studying many aspects of the exhaustible resource model described in Section 2.1. For the purpose of this paper, it suffices to mention that it forms the basis of the approach used in Mitra *et al*. (2013) to study the issue of

[3] Usually we employ the strict inequality for an interval of time, because in continuous time spikes do not matter. Here, however, $c(t)$ is continuous, so that we can use this equivalent definition.

sustainability as well as the issue of existence of a maximin path in the exhaustible resource model. This approach consists in defining an appropriate *resource requirement function* by using (HaR), and we briefly review this concept.

Let

$$D = \{(c, k) \in \mathbb{R}^2_{++} : \text{there exists } r > 0 \text{ such that } F(k, r) > c\}$$

be a domain set of consumption–capital pairs which allow for positive capital accumulation. Further, define the associated correspondence

$$D(k) = \{c \in \mathbb{R}_{++} : (c, k) \in D\}, \quad \text{for } k \in \mathbb{R}_{++}.$$

Note that for every $k > 0$, $D(k)$ is non-empty, since every $0 < c < F(k, 1)$ belongs to $D(k)$. In fact, it is easy to verify that $D(k)$ is the interval $(0, \lim_{r\to\infty} F(k, r))$.

The following lemma from Mitra *et al*. (2013) provides the key result on the resource requirement function $\mathbf{r}(c, k)$ defined on the domain D.

Lemma 1 *Assume that F satisfies assumptions **A1–A3**, and let* $(c, k) \in D$*. Then* $\mathbf{p}(c, k)$ *and* $\mathbf{r}(c, k)$*, defined by*

$$\begin{aligned} &(i)\ \mathbf{p}(c, k) = \min_{\{r>0:F(k,r)>c\}} \frac{r}{F(k, r) - c}, \\ &(ii)\ \mathbf{r}(c, k) = \arg\min_{\{r>0:F(k,r)>c\}} \frac{r}{F(k, r) - c}, \end{aligned} \tag{4}$$

for all $(c, k) \in D$*, are continuously differentiable single-valued functions from D to* $\mathbb{R}_{++}$ *satisfying*

$$\mathbf{p}_1(c, k) > 0 \text{ and } \mathbf{r}_1(c, k) > 0, \quad \text{for all } (c, k) \in D. \tag{5}$$

It follows from the lemma that Hartwick's rule, prescribing that (given $(c, k) \in D$) the value of resource depletion be reinvested in augmentable capital, is satisfied if and only if $r = \mathbf{r}(c, k)$. That is,

$$F(k, \mathbf{r}(c, k)) - c = F_2(k, \mathbf{r}(c, k))\mathbf{r}(c, k) \tag{6}$$

holds, and (given $(c, k) \in D$) there is no $r > 0$, distinct from $\mathbf{r}(c, k)$, for which (HaR) holds.

3 Review of basic results

We use this section to review some of the basic results which form a natural backdrop for the study of maximin value and policy functions. These results relate to the issues of (a) sustainability of a positive consumption level for the entire future; (b) the existence of a maximin path; (c) competitive price support properties of maximin paths. We take up each of these issues in turn in the following subsections.

3.1 Sustainability

A path $(c(t), k(t), r(t))$ from (k_0, m_0) *can sustain a positive consumption level* $c > 0$ if $c(t) \geq c$ for all $t \geq 0$. Assumptions **A1–A3** do not imply that there is a path $(c(t), k(t), r(t))$ from $(k_0, m_0) \gg 0$

which can sustain a positive consumption level. That is, the set

$$C(k_0, m_0) \equiv \{c \in \mathbb{R}_{++} : \text{ there is a path } (c(t), k(t), r(t)) \text{ from } (k_0, m_0) \text{ with } c(t) \geq c \text{ for } t \geq 0\}$$

of *positive sustainable consumption levels* need not be non-empty. In particular, if

$$F(k, r) = k^a r^b \quad \text{for } (k, r) \in \mathbb{R}^2_+, \text{ with } a > 0, b > 0 \text{ and } a + b \leq 1, \tag{7}$$

then assumptions **A1–A3** are clearly satisfied. However, as shown by Solow (1974, section 8 and appendix B), the set $C(k_0, m_0)$ is non-empty if and only if $a > b$.

For the more general class of production functions *F*, satisfying **A1–A3**, a complete characterization of technologies for which a positive consumption level can be sustained from every initial stock $(k_0, m_0) \gg 0$ has been provided in Mitra *et al.* (2013) by using the resource requirement function $\mathbf{r}(c, k)$, and the associated resource use per unit of capital accumulation function $\mathbf{p}(c, k)$, obtained in Lemma 1.

If $(c, k) \in D$, then by Lemma 1 there are $k' < k$ and $c' > c$ such that $(0, c') \times (k', \infty) \subset D$. Since $\mathbf{p}(c, \cdot)$ is continuous on (k', ∞), the Riemann integral $\int_k^{k''} \mathbf{p}(c, x)dx$ is well defined for every $k'' > k$. So, we can define $\mathbf{m} : D \to \mathbb{R}_{++} \cup \{+\infty\}$ by

$$\mathbf{m}(c, k) = \int_k^\infty \mathbf{p}(c, x)dx.$$

Since $\mathbf{p}(c, x)$ is the required resource input per unit of capital accumulation if the capital stock equals x, the function $\mathbf{m}$ determines the required cumulative resource input needed to sustain the consumption level c from the initial capital stock k. It enables one to obtain a technological characterization of the sustainability problem, without reference to a time path, as stated in the following result. [4]

Proposition 1 *Assume **A1–A3**, and let $(k_0, m_0) \in \mathbb{R}^2_{++}$. Then $C(k_0, m_0)$ is non-empty if and only if*

$$\inf_{c \in D(k_0)} \mathbf{m}(c, k_0) < m_0.$$

Remark 1 *It is easy to check that when the production function F has the Cobb–Douglas form, given by (7), we have $D = \mathbb{R}^2_{++}$ and $D(k) = (0, \infty)$ for each $k > 0$. Thus,* $\mathbf{r}$ *and* $\mathbf{p}$ *are functions from $\mathbb{R}^2_{++}$ to $\mathbb{R}_{++}$ given by*

$$\mathbf{r}(c, k) = \frac{c^{(1/b)}}{(1 - b)^{(1/b)} k^{(a/b)}} \quad \text{and} \quad \mathbf{p}(c, k) = \frac{c^{(1-b)/b}}{b(1 - b)^{(1-b)/b}} \cdot \frac{1}{k^{(a/b)}}.$$

Given any $(k_0, m_0) \in \mathbb{R}^2_{++}$, we have (for every $c > 0$) $\mathbf{m}(c, k_0) < \infty$ if and only if $a > b$, and

$$\inf_{c \in D(k_0)} \mathbf{m}(c, k_0) = \begin{cases} 0 & \text{if } a > b \\ \infty & \text{if } a \leq b. \end{cases}$$

By Proposition 1, $C(k_0, m_0)$ is non-empty if and only if $a > b$, which is the result of Solow (1974).

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[4] The Cass and Mitra (1991) integral criterion characterization of sustainability also translates information about time paths to information about the technology. However, while the Mitra *et al*. (2013) characterization focuses directly on maintaining constant consumption (by following Hartwick's rule), the Cass–Mitra characterization focuses on behavior associated with maintaining constant output as a means to providing a consumption stream that is bounded away from zero. The Cass–Mitra characterization is in a discrete-time model, where, as noted by Dasgupta and Mitra (1983), Hartwick's rule does not hold for efficient and egalitarian paths. So obeying Hartwick's rule is not a natural benchmark in that setting.

3.2 The existence of a maximin path

Under **A1–A3**, given any $(k_0, m_0) \in \mathbb{R}^2_+$, there always exists a maximin path; see Mitra *et al*. (2013, theorem 2). However, if $C(k_0, m_0)$ is empty, any maximin path is necessarily trivial. Since importance of a maximin path arises only when it is non-trivial, it is useful to state separately a result here regarding the existence of a non-trivial maximin path. This happens whenever $C(k_0, m_0)$ is non-empty; the technological conditions for that to happen have already been completely characterized in Proposition 1.

Proposition 2 *Assume **A1–A3**, and let $(k_0, m_0) \in \mathbb{R}^2_{++}$ with $C(k_0, m_0)$ non-empty. Then there exists a non-trivial maximin path $(c(t), k(t), r(t))$ from $(k_0, m_0) \in \mathbb{R}^2_{++}$. Further, the path $(c(t), k(t), r(t))$ can be chosen to satisfy:*

(i) $(k(t), r(t)) \gg 0$ *for* $t \geq 0$;
(ii) $c(t)$ *is a positive constant for all* $t \geq 0$;
(iii) $\dot{k}(t) = r(t)F_2(k(t), r(t))$ *for all* $t \geq 0$;
(iv) $\dot{k}(t) > 0$ *for all* $t \geq 0$.

Thus, the non-trivial maximin path $(c(t), k(t), r(t))$ can be chosen to be interior, egalitarian, and to satisfy (HaR) for all $t \geq 0$, and consequently to exhibit positive capital accumulation for all $t \geq 0$. The statement of Proposition 2 is slightly more explicit than the comparable statement of theorem 2 in Mitra *et al*. (2013), where a complete proof can be found.

3.3 Hotelling's rule and capital value transversality

A crucial part of our discussion of the maximin value function in Section 4 will depend on *competitive* or *shadow prices* associated with the maximin path of Proposition 2; at these prices, the capital and resource use choices are such that profit is maximized at each point in time. We present a self-contained discussion here.

The approach is to show that the maximin path of Proposition 2 is *competitive* in the sense that it satisfies *Hotelling's rule*:

$$\frac{\dot{F}_2(k(t), r(t))}{F_2(k(t), r(t))} = F_1(k(t), r(t)), \quad \text{for all } t \geq 0, \tag{8}$$

equating the returns on the capital good and the exhaustible resource.

As noted in Buchholz *et al*. (2005), Hotelling's rule can only be formulated when the marginal product of the resource is a differentiable function of time. The only general way of ensuring that this holds is to assume that, along the class of paths considered, $k(t)$ and $r(t)$ are differentiable functions of time, and to appeal to the chain rule of differentiation. However, in our definition of a path in Section 2, the only restriction imposed on $r(t)$ is that it be continuous as a function of t, and it is with this definition that Propositions 1 and 2 have been established in Mitra *et al*. (2013).

Assume **A1–A3**, and let $(k_0, m_0) \in \mathbb{R}^2_{++}$ with $C(k_0, m_0)$ non-empty. Let $(c(t), k(t), r(t))$ be the maximin path from (k_0, m_0) satisfying conditions (i)–(iv) of Proposition 2. To show that it also satisfies (8), we first show that $r(t)$ is a continuously differentiable function of t. Note that by (1)(i) and condition (iii) of Proposition 2:

$$r(t)F_2(k(t), r(t)) = F(k(t), r(t)) - c, \quad \text{for } t \geq 0,$$

where $c(t) = c$ for $t \geq 0$, as given by condition (ii) of Proposition 2. Thus $(c, k(t), r(t)) \gg 0$ satisfies (HaR) for each $t \geq 0$. Consequently,

$$r(t) = \mathbf{r}(c, k(t)), \quad \text{for each } t \geq 0, \tag{9}$$

where $\mathbf{r}$ is the function obtained in Lemma 1. By condition (iv) of Proposition 2, we have $F(k(t), r(t)) > c$, and so $(c, k(t)) \in D$ for each $t \geq 0$. By Lemma 1, the function $\mathbf{r}$ is continuously differentiable on D. Since $k(t)$ is a continuously differentiable function of t, we can infer from (9) and the chain rule that $r(t)$ is a continuously differentiable function of t.

We can now use Buchholz *et al*. (2005, proposition 3) to assert that (8) holds. We state this finding formally in the next result.

Proposition 3 *Assume **A1–A3**, and let $(k_0, m_0) \in \mathbb{R}^2_{++}$ with $C(k_0, m_0)$ non-empty. Let $(c(t), k(t), r(t))$ be a non-trivial maximin path from (k_0, m_0) satisfying conditions (i)–(iv), of Proposition 2. Then $r(t)$ and $F_2(k(t), r(t))$ are continuously differentiable functions of t for $t \geq 0$, and*

$$\frac{\dot{F}_2(k(t), r(t))}{F_2(k(t), r(t))} = F_1(k(t), r(t)), \quad \text{for all } t \geq 0. \tag{HoR}$$

We associate with the maximin path $(c(t), k(t), r(t))$ of Proposition 3 a path of *present value prices* $(p(t))$ as follows:

$$p(t) = 1/F_2(k(t), r(t)), \quad \text{for } t \geq 0. \tag{10}$$

Then, given the concavity of F, one can verify that (HoR) implies *profit maximization*; that is, for all $t \geq 0$, and all $(k, r) \in \mathbb{R}^2_+$, we have

$$\begin{aligned} \pi(k(t), r(t)) \equiv p(t)F(k(t), r(t)) - (-\dot{p}(t))k(t) - r(t) \geq p(t)F(k, r) \\ - (-\dot{p}(t))k - r \equiv \pi(k, r), \end{aligned} \tag{11}$$

where $(-\dot{p}(t))$ is to be interpreted as the rental rate on capital. Note that the present value price of the resource is a positive constant, and definition (10) ensures that this constant can be normalized to unity as indicated in (11).

The present value price path $(p(t))$, defined by (10), will play an important role in the next section, and we now formally record some of the useful properties pertaining to this price path.

Proposition 4 *Assume **A1–A3**, and let $(k_0, m_0) \in \mathbb{R}^2_{++}$ with $C(k_0, m_0)$ non-empty. Let $(c(t), k(t), r(t))$ be a non-trivial maximin path from (k_0, m_0) satisfying conditions (i)–(iv) of Proposition 2, and let $(p(t))$ be the price path associated with it, as defined by (10). Then the following conditions hold:*

(i) $\delta_t(k, r) \equiv \pi(k(t), r(t)) - \pi(k, r) \geq 0$ *for all* $(k, r) \in \mathbb{R}^2_+$ *and all* $t \geq 0$.
(ii) $\lim_{t\to\infty} p(t)k(t) = 0$.

(iii) If $(c'(t), k'(t), r'(t))$ *is any path from* $(k'_0, m'_0) \in \mathbb{R}^2_+$, *and* $T \geq 0$, *then*

$$\int_0^T p(t)(c'(t) - c)dt = -\int_0^T \delta_t(k'(t), r'(t))dt + p(0)(k'_0 - k_0) + (m'_0 - m_0) + p(T)(k(T) - k'(T)) + (m(T) - m'(T)). \quad (12)$$

Condition (i) of Proposition 4 is just a restatement of (11). Condition (ii) of Proposition 4 is known as the *capital value transversality condition*. This result can be found in Mitra *et al*. (2013, proposition 5); its proof is part of the proof of proposition 5 in that paper. Condition (iii) of Proposition 4 follows from fairly standard calculations. We provide a self-contained exposition below.

For all $t \geq 0$, we have

$$\begin{aligned} p(t)(c'(t) - c) &= [p(t)F(k'(t), r'(t)) - p(t)\dot{k}'(t)] - [p(t)F(k(t), r(t)) - p(t)\dot{k}(t)] \\ &= [p(t)F(k'(t), r'(t)) - (-\dot{p}(t))k'(t) - r'(t)] - [p(t)\dot{k}'(t) + \dot{p}(t)k'(t)] + r'(t) \\ &\quad - [p(t)F(k(t), r(t)) - (-\dot{p}(t))k(t) - r(t)] + [p(t)\dot{k}(t) + \dot{p}(t)k(t)] - r(t) \\ &= -\delta_t(k'(t), r'(t)) - \frac{d}{dt}[p(t)k'(t)] + \frac{d}{dt}[p(t)k(t)] + (r'(t) - r(t)). \end{aligned} \quad (13)$$

Integrating (13) from $t = 0$ to $t = T$, and rearranging terms, one gets (12).

4 Efficiency and uniqueness of maximin paths

It is an open question whether, under assumptions **A1–A3**, and with $(k_0, m_0) \in \mathbb{R}^2_{++}$ with $C(k_0, m_0)$ non-empty, a maximin path $(c(t), k(t), r(t))$ from (k_0, m_0) is necessarily efficient. In fact, it is not known whether even the maximin path satisfying conditions (i)–(iv) of Proposition 2 is necessarily efficient.

Under a reasonable assumption on the importance of the exhaustible resource in production, it turns out that any maximin path $(c(t), k(t), r(t))$ from (k_0, m_0) is necessarily efficient, and in fact there is only one maximin path from (k_0, m_0). This assumption considerably simplifies the presentation of the basic theory, by eliminating alternate scenarios. We state this assumption here, and proceed to maintain it in the next two sections of the paper.

(A4) $\beta \equiv \inf_{(k,r)\gg 0} \dfrac{rF_2(k, r)}{F(k, r)} > 0.$

The assumption states that the share of the resource in output is bounded away from zero. It was introduced into this literature in Mitra (1978).

Since the basic theory on the efficiency and uniqueness of maximin paths in the exhaustible resource model is not readily available, we provide below a reasonably self-contained treatment, including not only the key results but also the proofs of these results.

A preliminary implication of **A4** is that an increase in the initial resource stock always leads to an increased sustainable consumption level.

Lemma 2 *Assume **A1–A4**, and let* $(k_0, m_0) \in \mathbb{R}^2_{++}$ *with* $C(k_0, m_0)$ *non-empty. Let* $(c'(t), k'(t), r'(t))$ *be a path from* (k_0, m_0) *with* $\inf_{t\geq 0} c'(t) = c'$. *If* $\bar{m} > m_0$, *then there is a path* $(\tilde{c}(t), \tilde{k}(t), \tilde{r}(t))$ *from* $(k_0, \bar{m})$ *satisfying* $\inf_{t\geq 0} \tilde{c}(t) > c'$.

PROOF: Let $(c(t), k(t), r(t))$ be the non-trivial maximin path from (k_0, m_0) satisfying conditions (i)–(iv), whose existence is ensured by Proposition 2. Denote the constant level of consumption on this path by c; then $c \geq c'$. Define $(k''(t), r''(t)) = \frac{1}{2}(k(t), r(t)) + \frac{1}{2}(k'(t), r'(t))$, and $c''(t) = F(k''(t), r''(t)) - \dot{k}''(t)$ for $t \geq 0$. Then, using the concavity of F, we have $c''(t) \geq \frac{1}{2}c(t) + \frac{1}{2}c'(t) \geq c'$. Then $(c''(t), k''(t), r''(t))$ is a path from (k_0, m_0).

Note that $m_0 \geq \int_0^\infty r''(t)dt > 0$, and define

$$\lambda = \frac{\bar{m}}{\int_0^\infty r''(t)dt}.$$

Then $\lambda > 1$ and $\int_0^\infty \lambda r''(t)dt = \bar{m}$. Now, define $(\tilde{k}(t), \tilde{r}(t)) = (k''(t), \lambda r''(t))$ and $\tilde{c}(t) = F(\tilde{k}(t), \tilde{r}(t)) - \dot{k}''(t)$ for $t \geq 0$. Then

$$\begin{aligned}
\tilde{c}(t) - c''(t) &= F(k''(t), \lambda r''(t)) - F(k''(t), r''(t)) \\
&\geq F_2(k''(t), \lambda r''(t))[\lambda - 1]r''(t) \\
&= \frac{F_2(k''(t), \lambda r''(t))\lambda r''(t)}{F(k''(t), \lambda r''(t))} \cdot \frac{[\lambda - 1]F(k''(t), \lambda r''(t))}{\lambda} \\
&\geq [\beta(\lambda - 1)/\lambda]F(k''(t), \lambda r''(t)) \\
&\geq [\beta(\lambda - 1)/\lambda]F(k''(t), r''(t)) \\
&\geq [\beta(\lambda - 1)/\lambda]\left[\frac{1}{2}F(k(t), r(t)) + \frac{1}{2}F(k'(t), r'(t))\right] \\
&\geq [\beta(\lambda - 1)/2\lambda]F(k(t), r(t)) \\
&\geq [\beta(\lambda - 1)/2\lambda]c.
\end{aligned}$$

Thus, $(\tilde{c}(t), \tilde{k}(t), \tilde{r}(t))$ is a path from $(k_0, \bar{m})$, and

$$\inf_{t\geq 0} \tilde{c}(t) > \inf_{t\geq 0} c''(t) \geq \inf_{t\geq 0}[\tfrac{1}{2}c + \tfrac{1}{2}c'(t)] = \tfrac{1}{2}c + \tfrac{1}{2}\inf_{t\geq 0} c'(t) \geq c'. \qquad \square$$

Using Lemma 2, one can establish the important result that the particular maximin path obtained in Proposition 2 is necessarily efficient.

Proposition 5 *Assume **A1–A4**, and let $(k_0, m_0) \in \mathbb{R}^2_{++}$ with $C(k_0, m_0)$ non-empty. Let $(c(t), k(t), r(t))$ be a non-trivial maximin path from (k_0, m_0) satisfying conditions (i)–(iv), whose existence is ensured by Proposition 2. Then $(c(t), k(t), r(t))$ is efficient.*

PROOF: Denote the constant level of consumption on this path $(c(t), k(t), r(t))$ by c. If $(c(t), k(t), r(t))$ is inefficient, there is a path $(c'(t), k'(t), r'(t))$ from (k_0, m_0) satisfying $c'(t) \geq c$ for all $t \geq 0$ and $c'(t) > c$ for some $t \geq 0$. Define $(k''(t), r''(t)) = \frac{1}{2}(k(t), r(t)) + \frac{1}{2}(k'(t), r'(t))$, and $c''(t) = F(k''(t), r''(t)) - \dot{k}''(t)$ for $t \geq 0$. Then, using the concavity of F, we have $c''(t) \geq \frac{1}{2}c + \frac{1}{2}c'(t)$ for all $t \geq 0$. So $(c''(t), k''(t), r''(t))$ is a path from (k_0, m_0), and $c''(t) \geq c$ for all $t \geq 0$ and $c''(t) > c$ for some $t = \tau \geq 0$. Since $c''(t)$ is continuous in t, there is $\varepsilon > 0$ such that $c''(t) > c$ for all $t \in I \equiv [\tau, \tau + \varepsilon]$.

Define

$$\mu = \min_{t \in I}[c''(t) - c], \quad \nu = \min_{t \in I} r''(t), \quad \nu' = \max_{t \in I} r''(t),$$
$$M = \min_{t \in I} k''(t), \quad M' = \max_{t \in I} k''(t), \tag{14}$$

noting that continuity of the relevant variables in t ensures that these magnitudes are well defined. Let $J = [M, M'] \times [\nu/2, \nu']$, and define

$$Q = \max_{(k,r) \in J} F_2(k, r). \tag{15}$$

Choose $\delta > 0$, small enough so that

$$(i)\ \delta\varepsilon^2 \leq \nu/2, \text{ and } (ii)\ Q\delta\varepsilon^2 \leq \mu/2. \tag{16}$$

Now define $(\tilde{k}(t), \tilde{r}(t))$ as follows: $\tilde{k}(t) = k''(t)$ for all $t \geq 0$, $\tilde{r}(t) = r''(t)$ for all $t \notin I$, and $\tilde{r}(t) = r''(t) - \delta(t - \tau)(\tau + \varepsilon - t)$ for $t \in I$. Note that $\tilde{r}(t)$ is a continuous function of t for $t \geq 0$, and

$$\int_0^\infty \tilde{r}(t)dt \leq m_0 - \int_\tau^{\tau+\varepsilon} \delta(t - \tau)(\tau + \varepsilon - t)dt \equiv m_0 - \theta. \tag{17}$$

Define $\tilde{c}(t) = F(\tilde{k}(t), \tilde{r}(t)) - \dot{k}''(t)$ for $t \geq 0$. Then $\tilde{c}(t) = c''(t)$ for $t \notin I$. For $t \in I$, we have

$$\begin{aligned}\tilde{c}(t) &\geq F(k''(t), r''(t) - \delta\varepsilon^2) - F(k''(t), r''(t)) + c''(t)\\ &\geq c + \mu - [F(k''(t), r''(t)) - F(k''(t), r''(t) - \delta\varepsilon^2)]\\ &\geq c + \mu - [F_2(k''(t), R(t))\delta\varepsilon^2],\end{aligned}$$

where $r''(t) - \delta\varepsilon^2 \leq R(t) \leq r''(t)$ is given by the mean value theorem. Note that by (14) and (16)(i), $r''(t) - \delta\varepsilon^2 \geq \nu - \delta\varepsilon^2 \geq \nu/2$, and $r''(t) \leq \nu'$ for $t \in I$; thus, $R(t) \in [\nu/2, \nu']$ for $t \in I$. Also, by (14), $k''(t) \in [M, M']$ for $t \in I$. Thus, using (15) and (16)(ii),

$$\begin{aligned}\tilde{c}(t) &\geq c + \mu - [F_2(k''(t), R(t))\delta\varepsilon^2]\\ &\geq c + \mu - Q\delta\varepsilon^2 \geq c + \mu/2.\end{aligned}$$

So $(\tilde{c}(t), \tilde{k}(t), \tilde{r}(t))$ is a path from $(k_0, m_0 - \theta)$, with $\tilde{c}(t) \geq c$ for all $t \geq 0$. Now, using Lemma 2, there is a path $(\hat{c}(t), \hat{k}(t), \hat{r}(t))$ from (k_0, m_0) with $\inf_{t \geq 0} \hat{c}(t) > \inf_{t \geq 0} \tilde{c}(t) \geq c$. This contradicts the fact that $(c(t), k(t), r(t))$ is a maximin path from (k_0, m_0), and establishes the result. □

It is now possible to assert the efficiency of *every* maximin path from $(k_0, m_0) \gg 0$, with $C(k_0, m_0)$ non-empty.

Corollary 1 *Assume* ***A1–A4****, and let* $(k_0, m_0) \in \mathbb{R}^2_{++}$ *with* $C(k_0, m_0)$ *non-empty. Let* $(c'(t), k'(t), r'(t))$ *be any maximin path from* (k_0, m_0)*. Then:*

(i) $(c'(t), k'(t), r'(t))$ *is efficient;*
(ii) $(c'(t), k'(t), r'(t))$ *is egalitarian;*
(iii) $\lim_{t\to\infty} m'(t) = 0$.

PROOF: Let $(c(t), k(t), r(t))$ be a non-trivial maximin path from (k_0, m_0) satisfying conditions (i)–(iv), whose existence is ensured by Proposition 2. Denote by c the constant consumption level on this path.

(i) Since $(c'(t), k'(t), r'(t))$ is a maximin path from (k_0, m_0), we have $\inf_{t\geq 0} c'(t) = c$. If $(c'(t), k'(t), r'(t))$ is inefficient, then there is a path $(c''(t), k''(t), r''(t))$ from (k_0, m_0) with $c''(t) \geq c'(t)$ for all $t \geq 0$, and $c''(t) > c'(t)$ for some t. This implies that $c''(t) \geq c$ for all $t \geq 0$ and $c''(t) > c$ for some t. But then $(c(t), k(t), r(t))$ is inefficient, contradicting Proposition 5.
(ii) Since $(c'(t), k'(t), r'(t))$ is a maximin path from (k_0, m_0), we have $\inf_{t\geq 0} c'(t) = c$. If $(c'(t), k'(t), r'(t))$ is not egalitarian, then there is some $t \geq 0$ for which $c'(t) > c$. Thus, $c'(t) \geq c$ for all $t \geq 0$ and $c'(t) > c$ for some t. But then $(c(t), k(t), r(t))$ is inefficient, contradicting Proposition 5.
(iii) Suppose $\lim_{t\to\infty} m'(t) = \mu > 0$. Then $(c'(t), k'(t), r'(t))$ is a path from $(k_0, m_0 - \mu)$. Thus, by Lemma 2, there is a path $(c''(t), k''(t), r''(t))$ from (k_0, m_0) satisfying $\inf_{t\geq 0} c''(t) > \inf_{t\geq 0} c'(t)$. This contradicts the fact that $(c'(t), k'(t), r'(t))$ is a maximin path from (k_0, m_0). □

We are now in a position to state the main result of this section, which asserts the uniqueness of the maximin path from $(k_0, m_0) \gg 0$, with $C(k_0, m_0)$ non-empty, and summarizes the properties of this maximin path.

Theorem 1 *Assume* ***A1–A4****, and let* $(k_0, m_0) \in \mathbb{R}^2_{++}$ *with* $C(k_0, m_0)$ *non-empty. Then there is a unique maximin path* $(c(t), k(t), r(t))$ *from* (k_0, m_0)*.*

(A) This path satisfies properties (i)–(iv) of Proposition 2; further, $r(t)$ *and* $F_2(k(t), r(t))$ *are continuously differentiable functions of* t *for* $t \geq 0$*, and (HoR) is satisfied.*
(B) Associated with this path is a price path $(p(t))$*, defined by* $p(t) = [1/F_2(k(t), r(t))]$ *for* $t \geq 0$*, and satisfying properties (i)–(iii) of Proposition 4.*
(C) The path $(c(t), k(t), r(t))$ *is efficient and satisfies* $\lim_{t\to\infty} m(t) = 0$.

PROOF: Let $(c'(t), k'(t), r'(t))$ be any maximin path from (k_0, m_0), and let $(c(t), k(t), r(t))$ be a maximin path from (k_0, m_0), satisfying conditions (i)–(iv) of Proposition 2. Denote by $c > 0$ the constant consumption level on this path. We know that $(c(t), k(t), r(t))$ is efficient by Proposition 5. By Corollary 1, $(c'(t), k'(t), r'(t))$ is efficient and egalitarian, and so $c(t) = c'(t) = c$ for all $t \geq 0$. Further, by Corollary 1, $\lim_{t\to\infty} m'(t) = 0 = \lim_{t\to\infty} m(t)$.

By Proposition 3, $r(t)$ and $F_2(k(t), r(t))$ are continuously differentiable functions of t for $t \geq 0$, and the path $(c(t), k(t), r(t))$ satisfies (HoR). Associated with the path $(c(t), k(t), r(t))$ is a price path $(p(t))$, defined by $p(t) = [1/F_2(k(t), r(t))]$ for $t \geq 0$, and satisfying properties (i)–(iii) of Proposition 4. Using condition (iii) of Proposition 4, we get for all $T \geq 0$,

$$0 = \int_0^T p(t)(c'(t) - c(t))dt \leq -\int_0^T \delta_t(k'(t), r'(t))dt + p(T)k(T) + m(T). \tag{18}$$

Using conditions (i) and (ii) of Proposition 4, and $\lim_{t\to\infty} m(t) = 0$, we see that $\int_0^\infty \delta_t(k'(t), r'(t))dt$ is well defined, and (18) implies that

$$\int_0^\infty \delta_t(k'(t), r'(t))dt \le 0. \tag{19}$$

Using condition (i) of Proposition 4 again, (19) implies that

$$\delta_t(k'(t), r'(t)) = 0, \quad \text{for all } t \ge 0, \tag{20}$$

noting that $\delta_t(k'(t), r'(t))$ is a continuous function of t. Thus, for all $t \ge 0$, and all $(k, r) \in \mathbb{R}^2_+$, we have

$$p(t)F(k'(t), r'(t)) - (-\dot{p}(t))k'(t) - r'(t) \ge p(t)F(k, r) - (-\dot{p}(t))k - r. \tag{21}$$

We claim that $k'(t) > 0$ for $t \ge 0$. For if $k'(T) = 0$ for some $T \ge 0$, then $\dot{k}'(T) = -c < 0$, and since $\dot{k}'(t)$ is a continuous function of t, there is $\varepsilon > 0$ such that $\dot{k}'(t) < 0$ for $t \in [T, T+\varepsilon]$. But this implies that $k'(T+\varepsilon) - k'(T) < 0$, so that $k'(T+\varepsilon) < 0$, a contradiction. This establishes our claim. Using $(k, r) = (0, 0)$ in (21), and the fact that $(c(t), k(t), r(t))$ satisfies (HoR), we have

$$p(t)F(k'(t), r'(t)) \ge (-\dot{p}(t))k'(t) + r'(t) \ge (-\dot{p}(t))k'(t) = p(t)F_1(k(t), r(t))k'(t) > 0,$$

so that $r'(t) > 0$ for $t \ge 0$. Now using (21), we obtain for $t \ge 0$,

$$\begin{aligned} &(i)\ p(t)F_1(k'(t), r'(t)) = (-\dot{p}(t)),\\ &(ii)\ p(t)F_2(k'(t), r'(t)) = 1. \end{aligned} \tag{22}$$

Using (22)(ii), we know that $F_2(k'(t), r'(t))$ is a continuously differentiable function of t. So, differentiating (22)(ii) with respect to t, and using (22)(i),

$$\frac{\dot{F}_2(k'(t), r'(t))}{F_2(k'(t), r'(t))} = \frac{(-\dot{p}(t))}{p(t)} = F_1(k'(t), r'(t)), \quad \text{for all } t \ge 0. \tag{23}$$

Thus, $(c'(t), k'(t), r'(t))$ satisfies (HoR). Using (22)(ii) we can also infer, by using **A3** and the implicit function theorem, that $r'(t)$ is a continuously differentiable function of t. Thus, using Buchholz *et al.* (2005, theorem 1), we obtain the result that $(c'(t), k'(t), r'(t))$ satisfies (HaR). By Lemma 1, $r'(t) = \mathbf{r}(c, k'(t))$ for all $t \ge 0$. This means that $(k'(t))$ is a solution of the differential equation

$$\dot{x}(t) = F(x(t), \mathbf{r}(c, x(t))) - c, \quad \text{for } t \ge 0,\ x(0) = k_0,$$

and so is $(k(t))$. We can now use Hirsch and Smale (1974, theorem 1, p. 162) to conclude that $k'(t) = k(t)$ for all $t \ge 0$, and so $r'(t) = \mathbf{r}(c, k'(t)) = \mathbf{r}(c, k(t)) = r(t)$ for all $t \ge 0$. This means that $(c'(t), k'(t), r'(t))$ is the same maximin path as $(c(t), k(t), r(t))$. This establishes the uniqueness of a maximin path from (k_0, m_0). Now (A) follows from Propositions 2 and 3; (B) follows from Proposition 4. Finally, (C) follows from Corollary 1. □

5 The maximin value function

We come now to one of the central concerns of the paper, the maximin value function and its properties. As already noted in Section 3.2, under assumptions **A1–A3**, there exists a maximin path $(c(t), k(t), r(t))$ from every $(k_0, m_0) \in \mathbb{R}^2_+$. We can therefore define a *maximin value function* (referred to henceforth simply as the value function) as a function $V : \mathbb{R}^2_+ \to \mathbb{R}_+$ by

$$V(k, m) = \inf_{t \geq 0} c(t),$$

where $(c(t), k(t), r(t))$ is a maximin path from (k, m). We proceed to examine the properties of the value function under the further assumption **A4**, which was introduced in Section 4.

If $(k, m) \in \mathbb{R}^2_+$ and $C(k, m)$ is empty, then for every path $(c(t), k(t), r(t))$ from (k, m), we have $\inf_{t \geq 0} c(t) = 0$, and so $V(k, m) = 0$. If $(k, m) \in \mathbb{R}^2_+$ and $C(k, m)$ is non-empty, then there is a path $(c(t), k(t), r(t))$ from (k, m), with $\inf_{t \geq 0} c(t) = c > 0$. By **A1** we must have $(k, m) \gg 0$. Thus, all the results obtained in the previous two sections apply. In particular, there is a unique maximin path $(c(t), k(t), r(t))$ from (k, m), with $c(t) = c > 0$ for all $t \geq 0$, and $V(k, m) = c$.

In what follows, we assume that there is some $(k, m) \gg 0$ such that $C(k, m)$ is non-empty. Thus, there is a path $(c(t), k(t), r(t))$ from (k, m), with $\inf_{t \geq 0} c(t) = c > 0$. This actually ensures that $C(k', m')$ is non-empty for *every* $(k', m') \gg 0$. To see this, define $\lambda = (k'/k)$, $\mu = (m'/m)$, $\nu = \min\{\lambda, \mu, 1\}$, and $k'(t) = k' + \nu(k(t) - k)$, $r'(t) = \nu r(t)$ for $t \geq 0$. Then $k'(0) = k'$, and

$$\int_0^\infty r'(t)dt = \int_0^\infty \nu r(t)dt \leq \int_0^\infty \mu r(t)dt \leq \mu m = m'.$$

Further, $r'(t) \geq 0$ for $t \geq 0$, and $k'(t) = \lambda k - \nu k + \nu k(t) \geq \nu k(t) \geq 0$ for $t \geq 0$. Thus, defining $c'(t) = F(k'(t), r'(t)) - \dot{k}'(t)$, we have

$$c'(t) \geq F(\nu k(t), \nu r(t)) - \nu \dot{k}(t) \geq \nu F(k(t), r(t)) - \nu \dot{k}(t) = \nu c(t) \geq \nu c > 0$$

for all $t \geq 0$. This establishes that $C(k', m')$ is non-empty.

Using this information, we have:

$$V(k, m) \begin{cases} = 0 & \text{if } k = 0 \text{ or } m = 0 \\ > 0 & \text{if } (k, m) \gg 0. \end{cases} \tag{24}$$

By **A2**, the function V is concave and non-decreasing on $\mathbb{R}^2_+$.

We would now like to study the nature of the value function as the resource stock m varies. So we fix the initial capital stock $k_0 > 0$ and, suppressing the first argument of the value function, we write

$$v(m) = V(k_0, m), \quad \text{for all } m \geq 0.$$

Clearly, we have the following properties of v:

$$\begin{aligned} &(i)\ v(0) = 0; \\ &(ii)\ v \text{ is concave on } \mathbb{R}_+; \\ &(iii)\ v \text{ is increasing on } \mathbb{R}_+; \\ &(iv)\ v \text{ is continuous on } \mathbb{R}_{++}. \end{aligned} \tag{25}$$

Property (ii) follows from the concavity of V on $\mathbb{R}^2_+$, property (iii) follows from Lemma 2, and property (iv) follows from property (ii).

Since v is concave on $\mathbb{R}_+$, it has left- and right-hand derivatives at each $m > 0$. Given (25)(ii) and (25)(iii), these derivatives are positive on $\mathbb{R}_{++}$. These derivatives are crucial in measuring the gain in social welfare of an economy when there is an increase in the resource stock (because of a new discovery of a resource pool) or the loss of social welfare of an economy following the destruction of a part of its resource stock (because of a natural or human-made disaster). (Here, social welfare is measured using the maximin intertemporal objective function.) Our purpose is to obtain precise estimates of these derivatives in terms of (ideally) observable magnitudes like the prices associated with the maximin path.

Using the results of the previous sections, we first state and prove a property which bounds the integral of the present value prices associated with a maximin path.

Lemma 3 *Assume **A1–A4**, and let $(k_0, m_0) \in \mathbb{R}^2_{++}$ with $C(k_0, m_0)$ non-empty. Let $(c(t), k(t), r(t))$ be the maximin path from (k_0, m_0), with constant consumption $c > 0$ and associated price path $(p(t))$, as given by Theorem 1. Then*

$$\int_0^\infty p(t)dt < \infty. \tag{26}$$

Proof: Let $m'_0 > m_0$. Then $C(k_0, m'_0)$ is non-empty; let $(c'(t), k'(t), r'(t))$ be the maximin path from (k_0, m'_0), with constant consumption c', as given by Theorem 1. Using Proposition 4(iii), we have, for all $T \geq 0$,

$$\int_0^T p(t)(c' - c)dt \leq (m'_0 - m_0) + p(T)k(T) + m(T). \tag{27}$$

By Lemma 2, we have $(c' - c) > 0$, and so, for all $T \geq 0$,

$$\int_0^T p(t)dt \leq \frac{(m'_0 - m_0)}{(c' - c)} + \frac{[p(T)k(T) + m(T)]}{(c' - c)}. \tag{28}$$

Letting $T \to \infty$, and noting that $\lim_{T\to\infty} p(T)k(T) = 0$ and $\lim_{T\to\infty} m(T) = 0$ by Theorem 1, the limit of the left-hand side of (28) exists as $T \to \infty$, and

$$\int_0^\infty p(t)dt \leq \frac{(m'_0 - m_0)}{(c' - c)}. \tag{29}$$

This establishes (26). □

Lemma 3 enables us to establish a key result of this section, which places appropriate bounds on the left- and right-hand derivatives of the value function.

The heuristics of the formula derived in Proposition 6 (see (30)), and refined later in Theorem 2 (see (35)), can be seen as follows. Suppose there is an increase in the initial resource stock m_0 by $\varepsilon > 0$. One can then increase the resource flow at time t by $\Delta r(t) = \varepsilon p(t)/\int_0^\infty p(t)dt$ for all $t \geq 0$. (Notice how the result of Lemma 3 is crucial for this step.) Then $\int_0^\infty \Delta r(t)dt = \varepsilon$, so this is feasible to carry out. The increase in the resource flow will increase the output at each t, and this additional output can be entirely consumed, thus keeping the capital stock at each t the same as it was before the change in the resource stock. Then the increase in consumption at time t is approximately equal to $F_2(k(t), r(t))\Delta r(t)$. Thus, one can get a *uniform* consumption gain for all t approximately equal to $\varepsilon / \int_0^\infty p(t)dt$. This suggests that the right-hand derivative of the value function should be equal to $1/\int_0^\infty p(t)dt$.

Proposition 6 *Assume **A1–A4**, and let* $(k_0, m_0) \in \mathbb{R}^2_{++}$ *with* $C(k_0, m_0)$ *non-empty. Let* $(c(t), k(t), r(t))$ *be the maximin path from* (k_0, m_0)*, with constant consumption* $c > 0$ *and associated price path* $(p(t))$*, as given by Theorem 1. Then*

$$v'_-(m_0) \geq \frac{1}{\int_0^\infty p(t)dt} \geq v'_+(m_0). \tag{30}$$

PROOF: Let $m'_0 > m_0$. Then $C(k_0, m'_0)$ is non-empty; let $(c'(t), k'(t), r'(t))$ be the maximin path from (k_0, m'_0), with constant consumption c' as given by Theorem 1. By Lemma 2, we have $(c' - c) > 0$. Then, using the proof of Lemma 3, we get (29), which can be rewritten as

$$\frac{c' - c}{m'_0 - m_0} \leq \frac{1}{\int_0^\infty p(t)dt}. \tag{31}$$

Let $m'_0 \downarrow m_0$; then $c' \downarrow c$, and we get the right-hand inequality of (30).

Let $0 < m''_0 < m_0$. Then $C(k_0, m''_0)$ is non-empty; let $(c''(t), k''(t), r''(t))$ be the maximin path from (k_0, m''_0), with constant consumption c'', as given by Theorem 1. By Lemma 2, we have $(c - c'') > 0$. Using Proposition 4 (iii), we have, for all $T \geq 0$,

$$\int_0^T p(t)(c'' - c)dt \leq (m''_0 - m_0) + p(T)k(T) + m(T)$$

which can be rewritten as

$$\int_0^T p(t)(c - c'')dt \geq (m_0 - m''_0) - p(T)k(T) - m(T). \tag{32}$$

Letting $T \to \infty$, and noting that $\lim_{T\to\infty} p(T)k(T) = 0$ and $\lim_{T\to\infty} m(T) = 0$ by Theorem 1, we obtain

$$\frac{c - c''}{m_0 - m''_0} \geq \frac{1}{\int_0^\infty p(t)dt}, \tag{33}$$

using (26) from Lemma 3. Let $m''_0 \uparrow m_0$; then $c'' \uparrow c$, and we get the left-hand inequality of (30). □

5.1 Derivative of the value function

Since v is concave on $\mathbb{R}_+$, we know that v is differentiable on $\mathbb{R}_{++}$ except possibly on a countable set of points (see Roberts and Varberg 1973, p. 7, theorem C). It follows from Proposition 6 that, except at a countable set of values of $m \in \mathbb{R}_{++}$ we have

$$v'(m) = \frac{1}{\int_0^\infty p(t)dt}, \tag{34}$$

where $(c(t), k(t), r(t))$ is the maximin path from (k, m), with associated price path $(p(t))$, as given by Theorem 1.

This is the basic result relating the maximin value function with the competitive prices associated with the maximin path and measures the *marginal (maximin) welfare change from a change in the resource stock*. In this subsection we show that the value function is differentiable for all $m \in \mathbb{R}_{++}$, and so formula (34) in fact applies to *all* resource stocks $m \in \mathbb{R}_{++}$. Crucial to this demonstration is the fact that there is a unique maximin path for every $m \in \mathbb{R}_{++}$.[5]

Theorem 2 *Assume* ***A1–A4****, and let* $(k_0, m_0) \in \mathbb{R}^2_{++}$ *with* $C(k_0, m_0)$ *non-empty. Let* $(c(t), k(t), r(t))$ *be the maximin path from* (k_0, m_0)*, with constant consumption* $\hat{c} > 0$ *and with associated price path* $(p(t))$*, as given by Theorem 1. Then* v *is differentiable at every* $m > 0$*, and*

$$v'(m_0) = \frac{1}{\int_0^\infty p(t)dt}. \tag{35}$$

PROOF: We establish that the left-hand inequality in (30) is an equality. Suppose to the contrary that it is a strict inequality. Then we must have

$$\theta \equiv \int_0^\infty p(t)dt - \frac{1}{v'_-(m_0)} > 0. \tag{36}$$

Clearly, by (26) of Lemma 3, we can pick $T > 0$, such that

$$\int_T^\infty p(t)dt \le \theta/3. \tag{37}$$

Using (36) and (37), we obtain

$$\int_0^T p(t)dt - \frac{1}{v'_-(m_0)} = \left[\int_0^\infty p(t)dt - \frac{1}{v'_-(m_0)}\right] - \int_T^\infty p(t)dt \ge 2\theta/3. \tag{38}$$

Define $a = F(1, 2m_0/T)$, $K = \max\{1, k_0\}$, $I = [m_0/2, 2m_0]$, $J = [k_0, Ke^{aT}]$, the domain set $E = [0, T] \times J$, and the function $f : E \times I \to \mathbb{R}$ by

$$f(t, x, m) = F(x, \mathbf{r}(v(m), x)) - v(m), \quad \text{for all } (t, x, m) \in E \times I. \tag{39}$$

Then f is continuous on the non-empty compact set $E \times I$, and therefore there is $M > 0$ such that $|f(t, x, m)| \le M$ for all $(t, x, m) \in E \times I$. By Theorem 1, for each $m \in I$, the differential equation

$$\dot{x} = f(t, x, m), \ x(0) = k_0$$

[5] In a different context, a similar connection between uniqueness of the optimal path and differentiability of the value function has been noted by Dechert and Nishimura (1983).

has a unique solution ϕ_m on $[0, T]$; in fact, this solution $\phi_m(t) \equiv k^m(t)$ for $t \in [0, T]$, where $(c^m(t), k^m(t), r^m(t))$ is the unique maximin path from (k_0, m). Using Coddington and Levinson (1955, theorem 4.1, p. 58), $\phi_m \to \phi_{m_0}$ uniformly on $[0, T]$ as $m \to m_0$.

We have $(\hat{c}, k_0) \in D$, and so there is $0 < \sigma < \hat{c}$, such that $(c, k_0) \in D$ for all $c \in Q = [\hat{c} - \sigma, \hat{c} + \sigma]$. This implies in turn that D contains the non-empty compact set $Q \times J$. Since $\mathbf{r}$ is continuously differentiable on D, we can find $M_1 > 0$ and $M_2 > 0$ such that

$$|\mathbf{r}_1(c, k)| \leq M_1 \text{ and } |\mathbf{r}_2(c, k)| \leq M_2, \quad \text{for all } (c, k) \in Q \times J. \tag{40}$$

Given any $\varepsilon > 0$, we can find $0 < \gamma < m_0$ such that whenever $|m - m_0| < \delta$, we have $|v(m) - v(m_0)| < \min\{\sigma, \varepsilon/2M_1\}$ by continuity of v on $\mathbb{R}_{++}$, and $|k^m(t) - k(t)| < \varepsilon/2M_2$ for all $t \in [0, T]$ by uniform convergence of ϕ_m to ϕ_{m_0} on $[0, T]$. Thus, $(v(m), k^m(t)) \in Q \times J$ for all $t \in [0, T]$, and using the continuous differentiability of $\mathbf{r}$ on $Q \times J$, and (40), we get $|r(v(m), k^m(t)) - r(v(m_0), k(t))| < \varepsilon$ for all $t \in [0, T]$. This establishes that $r^m(t)$ converges uniformly to $r(t)$ on $[0, T]$ as $m \to m_0$.

Since $r(t)$ is continuous in t and positive for all $t \geq 0$, we can find $0 < q < q' < \infty$, such that $r(t) \in [q, q']$ for all $t \in [0, T]$. Defining $S = [q'/2, 2q]$, we note that $F_2(k, r)$ is continuously differentiable on the compact set $J \times S$, and so it is Lipschitz on this set. Since $k^m(t)$ converges uniformly to $k(t)$ on $[0, T]$ and $r^m(t)$ converges uniformly to $r(t)$ on $[0, T]$ as $m \to m_0$, given any $\eta > 0$, we can find $0 < \delta' < m_0$ such that whenever $|m - m_0| < \delta'$, we have $(k^m(t), r^m(t)) \in J \times S$ for all $t \in [0, T]$ and $|F_2(k^m(t), r^m(t)) - F_2(k(t), r(t))| < \eta$ for all $t \in [0, T]$. That is, $F_2(k^m(t), r^m(t))$ converges uniformly to $F_2(k(t), r(t))$ on $[0, T]$ as $m \to m_0$. Thus, we can find $\delta > 0$ such that, for $0 < m_0 - m \leq \delta$, we have

$$p^m(t) \geq p(t) - \theta/3T, \quad \text{for all } t \in [0, T], \tag{41}$$

where $(c^m(t), k^m(t), r^m(t))$ is the maximin path from (k_0, m), with associated price path $(p^m(t))$, as given by Theorem 1.

The inequality in (41) implies in particular that, fixing $m = m_0 - \delta > 0$, we have

$$\int_0^\infty p^m(t)dt \geq \int_0^T p^m(t)dt \geq \int_0^T p(t)dt - \theta/3. \tag{42}$$

Combining (38) and (42), we obtain

$$\int_0^\infty p^m(t)dt \geq \int_0^T p(t)dt - \theta/3 \geq \frac{1}{v'_-(m_0)} + \theta/3 > \frac{1}{v'_-(m_0)}. \tag{43}$$

However, by Proposition 6, we have

$$\frac{1}{v'_+(m)} \geq \int_0^\infty p^m(t)dt \tag{44}$$

so that, combining (43) and (44), we obtain $(1/v'_+(m)) > (1/v'_-(m_0))$, which means that $v'_+(m) < v'_-(m_0)$. But this contradicts the concavity of v, since $0 < m < m_0$, and establishes that the left-hand inequality in (30) is an equality. The proof that the right-hand inequality in (30) is an equality is similar. Together, they establish (35), and therefore the differentiability of v on $\mathbb{R}_{++}$. □

6 The maximin policy function

Maximin policy functions (referred to henceforth simply as policy functions) are defined to be functions $g : \mathbb{R}^2_{++} \to \mathbb{R}$ and $h : \mathbb{R}^2_{++} \to \mathbb{R}$ such that, given any $(k_0, m_0) \in \mathbb{R}^2_{++}$, the maximin path $(c'(t), k'(t), r'(t))$ from (k_0, m_0) satisfies the system of differential equations:

$$\begin{aligned} &\dot{k}'(t) = g(k'(t), m'(t)), \quad \text{for } t \geq 0, \\ &\dot{m}'(t) = h(k'(t), m'(t)), \quad \text{for } t \geq 0, \\ &(k'(0), m'(0)) = (k_0, m_0), \end{aligned} \tag{45}$$

where $(m'(t))$ is defined by

$$m'(t) = m_0 - \int_0^t r'(\tau) d\tau \quad \text{for } t \geq 0. \tag{46}$$

Given the policy functions, the decision-maker at date t looks only at the current stocks of capital and the exhaustible resource $(k(t), m(t))$, and can decide on the maximin actions of how much to invest in capital accumulation $(\dot{k}(t))$, and how much of the exhaustible resource to use $(-\dot{m}(t) = r(t))$. One does not need to know any of the history of how one arrived at the current configuration of stocks $(k(t), m(t))$. Such policy functions can be derived only in cases where the problem has a *stationary* structure; that is, at each date t, the problem to be solved for the remaining future looks the same.

We now demonstrate how policy functions can be defined in our framework. We continue to assume **A1–A4**, and to assume that $C(k, m)$ is non-empty for some $(k, m) \in \mathbb{R}^2_{++}$. As we have demonstrated at the beginning of Section 5 (just before noting some basic properties of the value function in (24)), this entails that $C(k, m)$ is non-empty for every $(k, m) \in \mathbb{R}^2_{++}$, so that Theorem 1 applies to every configuration of initial stocks $(k, m) \in \mathbb{R}^2_{++}$. We can then define

$$h(k, m) = -\mathbf{r}(V(k, m), k), \quad \text{for all } (k, m) \in \mathbb{R}^2_{++}. \tag{47}$$

Note that by Theorem 1, there is a unique maximin path $(c(t), k(t), r(t))$ from (k, m), and $c(t) = V(k, m) > 0$ for all $t \geq 0$, and $F(k, r(0)) = F(k(0), r(0)) = c(0) + \dot{k}(0) > c(0) = V(k, m)$. Thus, $(V(k, m), k) \in D$ for all $(k, m) \in \mathbb{R}^2_{++}$ and the function h is well defined. By Lemma 1, $h(k, m) = -\mathbf{r}(V(k, m), k) < 0$ for all $(k, m) \in \mathbb{R}^2_{++}$.

We can also define

$$g(k, m) = F(k, \mathbf{r}(V(k, m), k)) - V(k, m), \quad \text{for all } (k, m) \in \mathbb{R}^2_{++}. \tag{48}$$

Since $\mathbf{r}(V(k, m), k) > 0$ for all $(k, m) \in \mathbb{R}^2_{++}$, g is also well defined for all $(k, m) \in \mathbb{R}^2_{++}$. There is a unique maximin path $(c(t), k(t), r(t))$ from (k, m), and $c(t) = V(k, m) > 0$, $\dot{k}(t) > 0$ for all $t \geq 0$. Thus,

$$F(k, \mathbf{r}(V(k, m), k)) - V(k, m) = F(k(0), \mathbf{r}(c(0), k(0))) - c(0) = \dot{k}(0) > 0.$$

Now, let $(k_0, m_0) \in \mathbb{R}^2_{++}$, and let $(c'(t), k'(t), r'(t))$ be the unique maximin path from (k_0, m_0). We know that $c'(t)$ is a constant for all $t \geq 0$. Denote this by c'. Then $V(k_0, m_0) = c'$. Consider any $T > 0$. Then $V(k'(T), m'(T)) \geq c'$, since $(c''(t), k''(t), r''(t))$ defined by $(c''(t), k''(t), r''(t)) =$

$(c'(T+t), k'(T+t), r'(T+t))$ for $t \geq 0$ is a path from $(k'(T), m'(T))$. If there is a path $(\tilde{c}(t), \tilde{k}(t), \tilde{r}(t))$ from $(k'(T), m'(T))$ with $\inf_{t\geq 0} \tilde{c}(t) > c'$, then clearly $(\hat{c}(t), \hat{k}(t), \hat{r}(t))$ defined by $(\hat{c}(t), \hat{k}(t), \hat{r}(t)) = (c'(t), k'(t), r'(t))$ for $0 \leq t \leq T$, and $(\hat{c}(t), \hat{k}(t), \hat{r}(t)) = (\tilde{c}(t-T), \tilde{k}(t-T), \tilde{r}(t-T))$ for $t > T$ is a path from (k_0, m_0), and $(c'(t), k'(t), r'(t))$ is inefficient, in contradiction to the statement of Theorem 1. Thus, $V(k'(T), m'(T)) = c'$ for all $T > 0$.

Using Theorem 1, we also have $\dot{k}'(t) = r'(t)F_2(k'(t), r'(t))$ for $t \geq 0$, so that by Lemma 1, $r'(t) = \mathbf{r}(c', k'(t)) = \mathbf{r}(V(k'(t), m'(t)), k'(t)) = -h(k'(t), m'(t))$ for $t \geq 0$. Using (46), we also have $\dot{m}'(t) = -r'(t)$ for $t \geq 0$, so that

$$\dot{m}'(t) = h(k'(t), m'(t)), \quad \text{for all } t \geq 0. \tag{49}$$

Also, $\dot{k}'(t) = F(k'(t), r'(t)) - c' = F(k'(t), \mathbf{r}(V(k'(t), m'(t)), k'(t))) - V(k'(t), m'(t))$ for $t \geq 0$, so that

$$\dot{k}'(t) = g(k'(t), m'(t)), \quad \text{for all } t \geq 0. \tag{50}$$

Thus, the maximin path $(c'(t), k'(t), r'(t))$ from (k_0, m_0) satisfies the system of differential equations (45), where $(m'(t))$ is given by (46).

6.1 Explicit solution in the Cobb–Douglas case

In the case where the production function has the Cobb–Douglas form

$$F(k, r) = k^a r^b, \quad \text{for } (k, r) \in \mathbb{R}^2_+ \text{ with } a > 0, b > 0 \text{ and } a + b \leq 1,$$

and $a > b$, assumptions **A1–A4** are satisfied (assumption **A4** is satisfied with $\beta = b$), and $C(k, m)$ is non-empty for all $(k, m) \in \mathbb{R}^2_{++}$. Thus, the theory developed in this paper in Sections 4–6 applies in this case. The explicit form of the value function has been derived by Solow (1974):

$$V(k, m) = (1-b)(a-b)^{b/(1-b)} m^{b/(1-b)} k^{(a-b)/(1-b)}, \quad \text{for } (k, m) \in \mathbb{R}^2_+. \tag{51}$$

Given the explicit solution of the function $\mathbf{r}$ (in Remark 1),

$$\mathbf{r}(c, k) = \frac{c^{(1/b)}}{(1-b)^{(1/b)} k^{(a/b)}}, \quad \text{for all } (c.k) \in \mathbb{R}^2_{++} \equiv D, \tag{52}$$

it is easy to obtain the policy function h explicitly by using (47), (51), and (52) as follows:

$$h(k, m) = \frac{(a-b)^{1/(1-b)} m^{1/(1-b)}}{k^{(1-a)/(1-b)}}, \quad \text{for all } (k, m) \in \mathbb{R}^2_{++} \tag{53}$$

and the policy function g explicitly by using (48), (51), and (52) as follows:

$$g(k, m) = b(a-b)^{b/(1-b)} k^{(a-b)/(1-b)} m^{b/(1-b)}, \quad \text{for all } (k, m) \in \mathbb{R}^2_{++}. \tag{54}$$

Remark 2 *The explicit forms of h and g in (53) and (54) can be used to study the short-run dynamics as well as the long-run behavior of maximin paths, by analyzing the dynamical system (45) in (k, m) space. It is useful to compare this with the phase diagram analysis offered in Solow (1974, figure 2, p. 38).*

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